International Journal of Theoretical Physics, Vol. 38, No. 8, 1999

Energy Levels of Interacting Fields in a Box

J. A. Espichán Carrillo[1] and A. Maia, Jr.[2]

Received March 29, 1999

We study the influence of boundary conditions on energy levels of interacting fields in a box and discuss some consequences when we change the size of the box. In order to do this we calculate the energy levels of bound states of a scalar massive field χ interacting with another scalar field ϕ through the Lagrangian $\mathcal{L}_{\text{int}} = \frac{3}{2} g\phi^2\chi^2$ in a one-dimensional box on which we impose Dirichlet boundary conditions. We find that the gap between the bound states changes with the size of the box in a nontrivial way. For the case where the masses of the two fields are equal and for large box the energy levels of Dashen–Hasslacher–Neveu (DHN model) are recovered and we have a kind of boson condensate for the ground state. Below a critical box size $L \sim 2.93\ (2\sqrt{2}/M)$ the ground-state level splits, which we interpret as particle–antiparticle production under small perturbations of box size. Below other critical sizes, $L \sim (6/10)\ (2\sqrt{2}/M)$ and $L \sim 1.71\ (2\sqrt{2}/M)$, of the box, the ground state and first excited state merge in the continuum part of the spectrum.

1. INTRODUCTION

Consider a simple system of two interacting fields described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}M_\phi^2\phi^2 - \frac{\lambda}{4}\phi^4 + \frac{1}{2}(\partial_\mu\chi)^2 + \frac{1}{2}M_\chi^2\chi^2 - \frac{3g}{2}\phi^2\chi^2$$

where λ and g are coupling contants. In this work we study the influence of boundary conditions on energy levels of the field χ taking into account its interaction with field ϕ. From the above Lagrangian we derive the following equations of motion for both fields:

[1] Instituto de Física "Gleb Wathagin," University of Campinas (UNICAMP), 13.081-970, Campinas (SP), Brazil.
[2] Instituto de Matemática, University of Campinas (UNICAMP), 13.081-970, Campinas (SP), Brazil; e-mail: maia@ime.unicamp.br.

$$-\partial^{\mu}\partial_{\mu}\chi + M_{\chi}^{2}\chi - 3\,g\phi^{2}\chi = 0 \tag{1}$$

$$-\partial^{\mu}\partial_{\mu}\phi + M_{\phi}^{2}\phi - \lambda\phi^{3} = 0 \tag{2}$$

In Eq. (2) we have neglected the term $3g\phi\chi^2$, which can be interpreted as the backreaction of field χ on the mass term of ϕ. This can be achieved if we impose that $|\chi| << \min\{M_\phi/\sqrt{3}g\}$. Of course other regimes can be studied from Eq. (2) by adopting different approximations.

Static solutions of the classical equation of motion (2) are given by (Dashen *et al.*, 1974)

$$\phi(x) = \pm\frac{M_\phi}{\sqrt{\lambda}}\tanh\left(\frac{M_\phi x}{\sqrt{2}}\right)$$

These are kink solutions which connect two vacua at $x = \pm\propto$. In this work, unlike χ, the field ϕ is not changed by boundary conditions. So in our approximation they are transparent to ϕ. In a future work we will consider the most general case.

Substituting these solutions in Eq. (1), we obtain

$$-\partial^{\mu}\partial_{\mu}\chi + M_{\chi}^{2}\chi - 3\,\frac{g}{\lambda}\,M_{\phi}^{2}\tanh^{2}\left(\frac{M_\phi x}{\sqrt{2}}\right)\chi = 0$$

Since we are interested in stationary solutions, we can write $\chi(x) = e^{-i\omega t}\psi(x)$, where ω are energy eigenvalues and the previous equation reduces to

$$\frac{d^2}{dx^2}\,\psi(x) + \left(M_{\chi}^{2} + \omega^{2} - 3\,\frac{g}{\lambda}\,M_{\phi}^{2}\tanh^{2}\left(\frac{M_\phi x}{\sqrt{2}}\right)\right)\psi(x) = 0 \tag{3}$$

This equation is similar to the one from the DHN model (Dahsen *et al.*, 1974), which describe kinks in (1 + 1) dimensions, but here two different mass parameters appear in the potential.

In the Section 2, we calculate the bound energy levels of the field χ (or ψ) constrained to a box in (1 + 1) dimensions with Dirichlet boundary conditions. We discuss how we can interpret the splitting of the ground state when the box is shrunk below a critical value as particle production from a vacuum condensate.

2. BOUND SPECTRUM

In this section we obtain the energy levels of the field χ by imposing Dirichlet boundary conditions. This is done as follows. Changing of variables $z = M_\phi x/\sqrt{2}$ and $\omega^2 = (\varepsilon - 2)M_\phi^2/2$, we write Eq. (3) as

$$\frac{d^2}{dz^2}\psi(z) + \left(2\frac{M_\chi^2}{M_\phi^2} + \varepsilon - 2 - 6\frac{g}{\lambda}\tanh^2 z\right)\psi(z) = 0$$

Making a new change of variable, namely $E = 2\beta + \varepsilon - 2$, where we have defined the mass ratio $\beta \equiv M_\chi^2/M_\phi^2$, the above equation reduces to

$$\frac{d^2}{dz^2}\psi(z) + \left(E - 6\frac{g}{\lambda}\tanh^2 z\right)\psi(z) = 0 \tag{4}$$

For the discrete (bound) spectrum case, $0 \leq \omega^2 < 2M_\chi^2$, we have that $2\beta \leq E < 6\beta$.

In order to find the corresponding solution of Eq. (4), we make the following variable dependent transformation (Morse and Fesbach, 1953):

$$\psi(z) = \operatorname{sech}^k(z)Y(z) \tag{5}$$

where the parameter $k \in \mathbf{R}$ will be determined below.

Substituting (5) in (4), we get an equation for $Y(z)$:

$$\frac{d^2Y}{dz^2} - 2k\tanh(z)\frac{dY}{dz} + \left[\frac{k^2 + E - 6g/\lambda}{\operatorname{sech}^2(z)} + 6\frac{g}{\lambda} - k^2 - k\right]\operatorname{sech}^2(z)Y(z) = 0 \tag{6}$$

In this work, we discuss the particular case $g = \lambda$. The general case, including an asymptotic study for weak and strong coupling constant g, will be presented elsewhere. Nevertheless the above condition leads to interesting results. First it is possible to turn this equation into a hypergeometric differential equation. This is done as follows:

1. We set the term dependent on the variable z in square brackets equal to zero, which gives a relation between k and E, i.e.,

$$k = \pm\sqrt{6 - E} \tag{7}$$

2. Making a change of the independent variable, namely,

$$\mu = \frac{1}{2}(1 - \tanh z) \tag{8}$$

we obtain a hypergeometric differential equation

$$\mu(1-\mu)\frac{d^2Y}{d\mu^2} + (k + 1 - 2(k+1)\mu)\frac{dY}{d\mu} - (k+3)(k-2)Y(\mu) = 0 \tag{9}$$

This equation has regular singular points at $\mu = 0$, $\mu = 1$, and $\mu = \infty$ and parameters $a = k + 3$, $b = k - 2$, and $c = k + 1$. The two independent analytic solutions are given by (Gradshteyn and Ryzhik, 1980)

$$Y_1(\mu) = {}_2F_1(k + 3, k - 2; k + 1; \mu) \tag{10}$$

$$Y_2(\mu) = \mu^{-k}\, {}_2F_1(-2, 3; 1 - k; \mu) \tag{11}$$

As is well known, if the parameter c is a positive integer, a solution of Eq. (9) will be $Y_1(\mu)$ and, if c is a negative integer, it will be given by $Y_2(\mu)$. There are other solutions containing a logarithmic term (Fuchs' Theorem). If c is a noninteger number, the set of two solutions $Y_1(\mu)$ and $Y_2(\mu)$ is a system of linearly independent solutions (Butkov, 1968). Next we study these three different cases for c. From now on, in order to simplify the notation, we denote the hypergeometric function ${}_2F_1$ simply as F.

Case I. *c* Is a Positive Integer

In this case, $c = k + 1 = n$, where $n = 1, 2, 3, \ldots$, and by Fuchs' Theorem (Butkov, 1968), the general solution is given by

$$Y(\mu) = AY_1(\mu) + BY_1(\mu)\ln|\mu| + B\sum_{s=0}^{\infty} a_s(-k)\mu^{s-k}$$

where A and B are arbitrary constants.

It is easy to see that this solution has a logarithmic divergence at $\mu = 0$. Since we want the solution $Y(\mu)$ be bounded, we must take $B = 0$. This comes from the constraint that for a large box ($L \to \infty$) we must recover DHN's kink and its bounded energy levels. So the solution reduces to

$$Y(\mu) = AF(n + 2, n - 3; n; \mu)$$

where k was substituted by $n - 1$.

Using the relations (5) and (7), as well as the change of variable $z = M_\phi x/\sqrt{2}$ in the above expression, we obtain

$$\psi(x) = A\,\mathrm{sech}^{(n-1)}\left(\frac{M_\phi x}{\sqrt{2}}\right) F\left(n + 2, n - 3; n; \frac{1}{2}\left[1 - \tanh\left(\frac{M_\phi x}{\sqrt{2}}\right)\right]\right) \tag{12}$$

Below, in order to simplify the notation, we will denote M_ϕ as M.

Now we impose Dirichlet boundary conditions at $x = \pm L/2$ for the solution (12), that is,

$$\psi\left(\mp\frac{L}{2}\right) = A\,\text{sech}^{(n-1)}\left(\frac{ML}{2\sqrt{2}}\right)F\left(n+2, n-3; n; \frac{1}{2}\left\{1 \pm \tanh\left(\frac{ML}{2\sqrt{2}}\right)\right\}\right)$$
$$= 0$$

From these relations we get the condition

$$F\left(n+2, n-3; n; \frac{1}{2}\left\{1 \pm \tanh\left(\frac{ML}{2\sqrt{2}}\right)\right\}\right) = 0 \qquad (13)$$

On the other hand, since $2\beta \leq E < 6\beta$, from Eq. (7), we can show that the parameter k satisfies the inequalities

$$\sqrt{6(1-\beta)} < k \leq \sqrt{6-2\beta}, \qquad -\sqrt{6-2\beta} \leq k < -\sqrt{6(1-\beta)}$$

Since we have considered that $k \in \mathbf{R}$, then from the above relations we obtain $\beta \leq 1$.

So the possible values of parameter k are in the intervals $0 < k \leq \sqrt{6} \sim 2.44$ or $2.44 \sim -\sqrt{6} \leq k < 0$. As in this case c is a positive integer, we must take the interval $0 < k \leq \sqrt{6}$.

The allowed integer values of k in the interval $0 < k \leq \sqrt{6}$ and the corresponding values of E [from Eq.(13)] are (using the relation $k = n - 1$)

$$\text{for} \quad n = 2, \quad k = 1: \qquad \text{then } E = 5$$
$$\text{for} \quad n = 3, \quad k = 2: \qquad \text{then } E = 2 \qquad (14)$$

The hypergeometric function in (13) can be written as Jacobi polynomials (Abramowitz and Stegun, 1972) and it is not difficult to prove that substituting the allowed values of k from (14) into (13), we do not get any consistent solution.

Case II. *c* Is a Negative Integer

Now we consider $c = k + 1 = -n$, where $n = 1, 2, 3, \ldots$, and then we must consider the interval $-\sqrt{6} \leq k < 0$. The solution of the hypergeometric equation in this case is given by $Y_2(\mu)$. As for the previous case, by Fuchs' Theorem we have that the general solution is given by

$$Y(\mu) = AY_2(\mu) + BY_2(\mu)\ln|\mu| + B\sum_{s=0}^{\infty} a_s(-k)\mu^{s-k}$$

where A and B are arbitrary constants.

In this case we can not discard the B terms in the same way as in the previous case, since for $\mu \to 0$ we have $\mu^{-k}\ln\mu \to 0$ and $s - k > 0$ for $s = 0, 1, 2, \ldots$. So we do not have a singularity at $\mu = 0$. Nevertheless, we notice that the relation $Y(\mu)/Y_2(\mu)$ is divergent in the asymptotic limit of $\mu = 0$, i.e.,

$$\lim_{\mu\to 0} \frac{Y(\mu)}{Y_2(\mu)} \to \propto$$

Again, we can impose our natural boundary condition, that is, for a very large box ($L \to \propto$), the DHN solution (Dashen *et al.*, 1974) should be recovered. In order to do this, we must impose the following asymptotic condition:

$$\lim_{\mu\to 0} \frac{Y(\mu)}{Y_2(\mu)} \to 1$$

In order for this condition be valid, the coefficient B must vanish. Therefore, our solution reduces to

$$Y(\mu) = A\mu^{(n+1)} F(-2, 3; 2 + n; \mu)$$

where k was substituted by $-(n + 1)$, and $A \neq 0$.

As we did before, using the relations (5) and (7), as well as the change of variable $z = Mx/\sqrt{2}$ in the above expression, we can write the solution in the original variables:

$$\psi(x) = A \operatorname{sech}^{-(n+1)}\left(\frac{Mx}{\sqrt{2}}\right) \frac{\{1 - \tanh(Mx/\sqrt{2})\}^{(n+1)}}{2^{(n+1)}} \times F\left(-2, 3; 2 + n; \frac{1}{2}\left\{1 - \tanh\left(\frac{Mx}{\sqrt{2}}\right)\right\}\right) \tag{15}$$

Now, imposing Dirichlet boundary conditions at $x = \pm L/2$, namely

$$\psi\left(\mp\frac{L}{2}\right) = A \operatorname{sech}^{-(n+1)}\left(\frac{ML}{2\sqrt{2}}\right) \frac{\{1 \pm \tanh(ML/\sqrt{2})\}^{(n+1)}}{2^{(n+1)}} \times F\left(-2, 3; 2 + n; \frac{1}{2}\left\{1 \pm \tanh\left(\frac{ML}{2\sqrt{2}}\right)\right\}\right) = 0$$

we obtain the condition

$$F\left(-2, 3; 2 + n; \frac{1}{2}\left\{1 \pm \tanh\left(\frac{ML}{2\sqrt{2}}\right)\right\}\right) = 0 \tag{16}$$

In this case, the allowed integer values of k, in the interval $-\sqrt{6} \leq k < 0$ are given by using the relation $k = -(n + 1)$,

$$\text{for} \quad n = 0, \quad k = -1: \qquad \text{then } E = 5$$

$$\text{for} \quad n = 1, \quad k = -2: \qquad \text{then } E = 2 \tag{17}$$

As in the previous case it is not difficult to write the above hypergeometric function (16) as Jacobi polynomials (Abramowitz and Stegun, 1972). A quick analysis shows that no consistent solution exists for finite $L \neq 0$. For $L = \infty$, the DHN case ($n = 0$) is obtained.

Case III. *c* Is a Noninteger

Since c is a noninteger number (positive or negative), the general solution is given by

$$Y(\mu) = AF(k + 3, k - 2; k + 1; \mu) + B(\mu)^{-k} F(-2, 3; 1 - k; \mu) \tag{18}$$

Now we repeat the previous steps, i.e., we use the relations (5) and (7), as well the change of variable $z = Mx/\sqrt{2}$ in the previous expression, and obtain

$$\psi(x) = \operatorname{sech}^k\left(\frac{Mx}{\sqrt{2}}\right)\left(AF\left(k + 3, k - 2; k + 1; \frac{1}{2}\left\{1 - \tanh\left(\frac{Mx}{\sqrt{2}}\right)\right\}\right)\right.$$
$$\left. + B \cdot 2^k\left[1 - \tanh\left(\frac{Mx}{\sqrt{2}}\right)\right]^{-k} F\left(-2, 3; 1 - k; \frac{1}{2}\left\{1 - \tanh\left(\frac{Mx}{\sqrt{2}}\right)\right\}\right)\right) \tag{19}$$

Now we must determine k and then, using relation (7), find the allowed values of E. The Dirichlet boundary conditions at $x = \pm L/2$ are given by

$$\psi\left(\mp\frac{L}{2}\right) = \operatorname{sech}^k\left(\frac{ML}{2\sqrt{2}}\right)\left(AF\left(k + 3, k - 2; k + 1; \frac{1}{2}\left\{1 \pm \tanh\left(\frac{ML}{2\sqrt{2}}\right)\right\}\right)\right.$$
$$\left. + B \cdot 2^k\left[1 \pm \tanh\left(\frac{ML}{2\sqrt{2}}\right)\right]^{-k} F\left(-2, 3; 1 - k; \frac{1}{2}\left\{1 \pm \tanh\left(\frac{ML}{2\sqrt{2}}\right)\right\}\right)\right) = 0$$

From these relations we obtain

$$AF\left(k + 3, k - 2; k + 1; \frac{1}{2}\left\{1 \pm \tanh\left(\frac{ML}{2\sqrt{2}}\right)\right\}\right)$$
$$+ B \cdot 2^k\left[1 \pm \tanh\left(\frac{ML}{2\sqrt{2}}\right)\right]^{-k} F\left(-2, 3; 1 - k; \frac{1}{2}\left\{1 \pm \tanh\left(\frac{ML}{2\sqrt{2}}\right)\right\}\right) = 0$$

This is a system of homogeneous equations for A and B. So this system has a nontrivial solution only if the determinant of system is zero, i.e.,

$$\left\{1-\tanh\left(\frac{ML}{2\sqrt{2}}\right)\right\}^{-k} F\left(k+3, k-2; k+1; \frac{1}{2}\left\{1+\tanh\left(\frac{ML}{2\sqrt{2}}\right)\right\}\right)$$
$$\times F\left(-2, 3; 1-k; \frac{1}{2}\left\{1-\tanh\left(\frac{ML}{2\sqrt{2}}\right)\right\}\right) - \left\{1+\tanh\left(\frac{ML}{2\sqrt{2}}\right)\right\}^{-k}$$
$$\times F\left(k+3, k-2; k+1; \frac{1}{2}\left\{1-\tanh\left(\frac{ML}{2\sqrt{2}}\right)\right\}\right)$$
$$\times F\left(-2, 3; 1-k; \frac{1}{2}\left\{1+\tanh\left(\frac{ML}{2\sqrt{2}}\right)\right\}\right) = 0 \tag{20}$$

Again we can write the above hypergeometric functions as Jacobi polynomials (Abramowitz and Stegun, 1972). In this way, from (20) we obtain a transcendental equation for the parameter k,

$$\left(\frac{1+\tanh(ML/2\sqrt{2})}{1-\tanh(ML/2\sqrt{2})}\right)^{k} = \pm\frac{k^2-1+3\,k\tanh(ML/2\sqrt{2})+3\tanh^2(ML/2\sqrt{2})}{k^2-1-3\,k\tanh(ML/2\sqrt{2})+3\tanh^2(ML/2\sqrt{2})} \tag{21}$$

Notice that substituting k by $-k$ in Eq. (21), it can be verified that this equation is satisfied. So these solutions are valid in the intervals $(0, 1) \cup (1, 2) \cup (2, \sqrt{6}) \subset \mathbf{R}$ and $(-\sqrt{6}, -2) \cup (-2, -1) \cup (-1, 0) \subset \mathbf{R}$. Observe that for the negative sign, $k = -1$ is a solution of Eq. (21), but it is not allowed since now we are considering $k \notin \mathbf{Z}$.

An analytic solution for the transcendental equation (21) was not found, but it is possible to obtain numerical solutions for it, as we can see in Fig. 1 for the case that $\beta = 1$.

Figure 1 shows the relation between parameter k and the size L of the box. Observe that both ground state and the first excited state shift with the size of the box. Starting with a very large box size L, as it decreases, the values of ω_0 and ω_1 increase from $\omega_0 = 0$ ($k = -2$ and $L \to \infty$) to $\omega_0 = \sqrt{2}M$ ($k = 0$), and from $\omega_1 = \sqrt{3/2}M$ ($k = -1$ and $L \to \infty$) to $\omega_1 = \sqrt{2}M$ ($k = 0$). Close to the critical value $\sim$0.6 for ω_0 and $\sim$1.71 for ω_1 these bound states merge in the continuum part of the spectrum ($k = 0$). On the other hand, for $L \to \infty$ the values of ω_0 and ω_1 decreases until they reach their minimum values $\omega_0 = 0$ and $\omega_1 = \sqrt{3/2}\,M$, respectively, just as in the DHN model. This behavior happens for both cases of positive and negative signs in Eq. (21) with negative k. For any positive k and large box size L the Eq. (21) has no solution, which can be checked analytically (see also

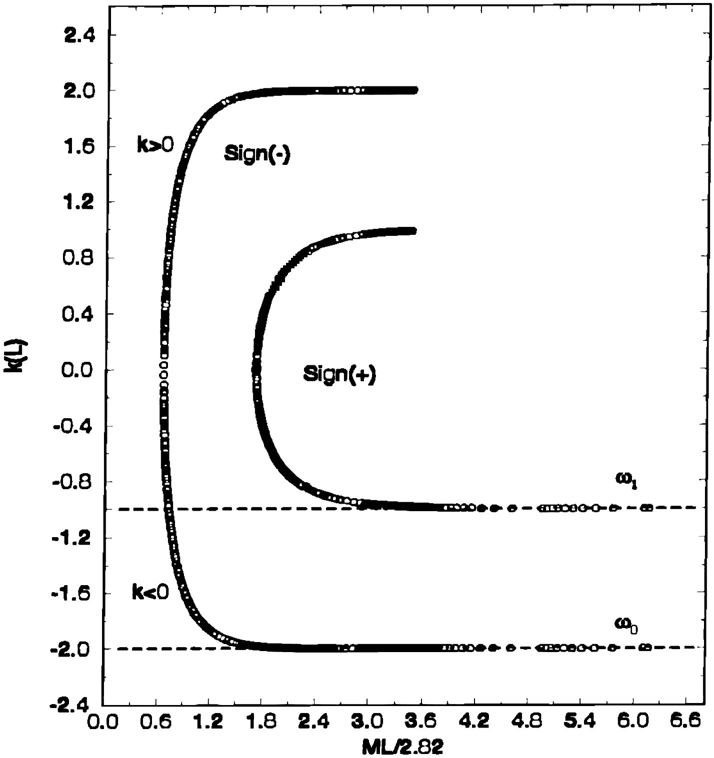

Fig. 1. Shifts of bound states with the size of the box. Here $\beta = 1$. The horizontal lines show the asymptotic values of the DHN model (Dashen *et al.*, 1974).

Fig. 1). So we discard positive values of k since for a large box ($L \to \infty$) they do not lead to DHN energy levels.

We can also obtain a relation between $\omega(L)/M$ and box size L. From Eq. (7) and the relation $\omega^2 = (\varepsilon - 2)M^2/2$ we get

$$\frac{\omega}{M} = \pm \sqrt{\beta - \beta - \frac{k^2(L)}{2}} \tag{22}$$

In Fig. 2 we plot two cases for this relation, namely $\beta = 1$ and $\beta = \frac{1}{2}$. For the first case ($\beta = 1$), the ground state ω_0 for large box size may form an approximated vacuum particle–antiparticle condensate, since both levels are exponentially close to one another. Roughly speaking, below a critical size ($\sim$2.93) any perturbation of box size will induce pair formation from this energy level. This could be important for particle production in the presence of strong fields. Squeezed fields could have an enhanced production.

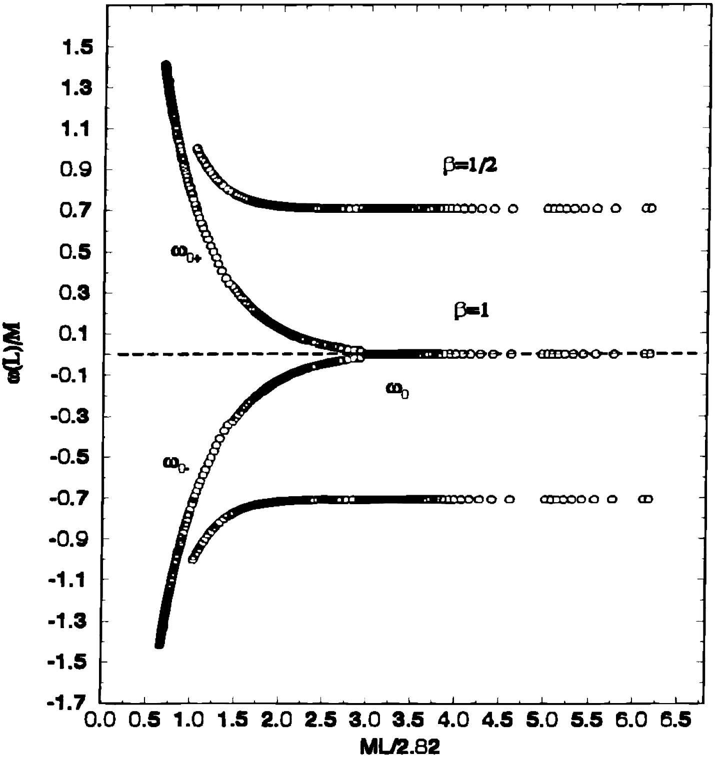

Fig. 2. Energy levels of the ground state of the field χ (or ψ) as a function of the box size L. The horizontal lines show the asymptotic values of the DHN model (Dashen *et al.*, 1974).

Another interesting behavior of the energy levels is shown in Fig. 3. We plot the difference between the levels ω_0 and ω_1 for arbitrary distance L. It is interesting to see a peak around the critical value $ML/2\sqrt{2} \sim 2.93$ and that the increasing part of the curve (left-hand side) from the peak shows a nonsmooth growth with several secondary maxima and minima.

3. CONCLUSIONS

In this work we calculated the solutions of a Klein–Gordon type equation in a one-dimensional box for which we impose Dirichlet boundary conditions and in the presence of a kink-type potential generated by a second scalar, self-interacting field which in our approximation is not subjected to the boundary conditions. Energy levels of bound states for nontrivial solutions are obtained as roots of a transcendental equation involving L and k. Although we have not obtained analytic solutions to it, we have studied numerical solutions (see Fig. 1).

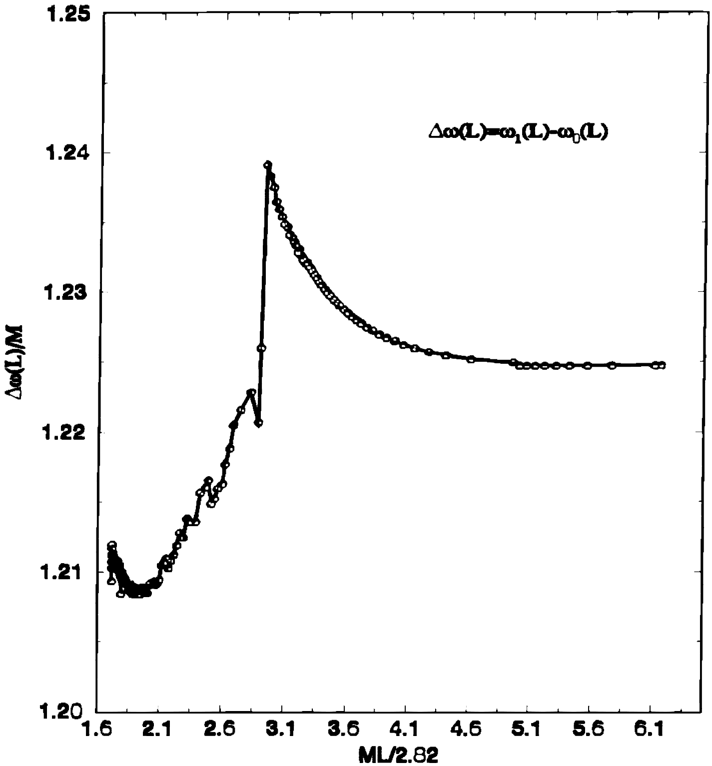

Fig. 3. The gap between levels ω_0 and ω_1 versus the size L of the box.

The ground state ω_0 and the first excited state ω_1 shift with the size L of the box (see Fig. 1) for both cases of positive and negative signs. As the size L of the box decreases, ω_0 increases in the interval $[0, \sqrt{2}M]$ and ω_1 in $[\sqrt{\frac{3}{2}}M, \sqrt{2}M]$. Close to the critical value ~ 0.6 for ω_0 and ~ 1.71 for ω_1, all the bounded states merge in the continuum part of the spectrum. For large distances ($L = \infty$) we obtain the energy levels of the DHN model. The decrease of L induces shifts on the bound-state levels of the system, and close to a critical size ~ 2.93 we have "just barely bound" condensate (Morse and Feshbach, 1953) that may decay against a small perturbation on the system, with particle pair creation (Fig. 2).

The gap between the two bound states presents a peak at $ML/2\sqrt{2} \sim 2.93$ and shows a nonsmooth behavior. It is interesting that the critical value for the splitting of the levels coincides with the value of the peak position, but we do not have any explanation for this fact. This, as well the study of the system taking into account finite boundary conditions on both fields ϕ and κ, will be dealt with elsewhere.

ACKNOWLEDGMENTS

This work was supported in part by FAPESP (Fundação de Amparo à Pesquisa do Estado de São Paulo), Brazil. The authors are indebted to Prof. A. A. Grib for his kind suggestions and discussions.

REFERENCES

Abramowitz, M., and Stegun I. A. (1972). *Handbook of Mathematical Functions*, Dover, New York.

Butkov, E. (1968). *Mathematical Physics*, Addison-Wesley, Reading, Massachusetts.

Dashen, R., Hasslacher, B., and Neveu, A. (1974). *Phys. Rev. D* **10**, 4131.

Gradshteyn, I. S., and Ryzhik, I. H. (1980). *Table of Integrals, Series and Products*, Academic Press, New York.

Morse, P. M., and Feshbach, H. (1953). *Methods of Theoretical Physics*, McGraw-Hill, New York.